NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-764

Some Existence and Sufficient Conditions of Optimality

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(NASA-CR-146576) SOME EXISTENCE AND SUFFICIENT CONDITIONS OF OPTIMALITY (Jet Propulsion Lab.) 24 p HC $3.50 CSCL 12B

N76-20929

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PASADENA, CALIFORNIA

February 15, 1976

PREFACE

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.

CONTENTS

ABSTRACT

Some existence and sufficient conditions of current interest in the field of optimal-control theory are discussed. This report briefly describes the role of existence and sufficiency conditions in the field of optimal control. The existence theorems are discussed for general nonlinear systems. However, the sufficient conditions pertain to "nearly" linear systems with integral convex costs. Moreover, a brief discussion of linear systems with multiple-cost functions is presented.

I. INTRODUCTION

The problem of automatic control has been discussed for many years, and various techniques and results have been developed to handle control problems. Within the last two decades, such problems have become of much greater interest due to the development of new control devices and the feasibility of implementation via digital computers. Although we shall discuss the optimal-control problem from a general point of view, where the cost functions, the constraints, and the target sets are quite general, some special examples will also be presented. Some of the typical problems of optimal-control problems are:

(1) Minimization of the fuel consumption of a given vehicle from a given initial state to a target state.

(2) Minimization of the time to transfer a given state to a final state.

(3) Minimization of energy.

Modern control theory frequently utilizes very complex systems, and, therefore, it is often impossible to determine an optimal control experimentally without a complete theoretical investigation. In the optimal-control theory, the physical system is often characterized by a state variable differential equation model; it may not, in general, represent the intrinsic physical reality that it purports to describe. In these situations, the optimal solution may not indeed exist, although one may intuitively feel that the solution should exist. If the optimal control does not exist, it can be implied that there is something degenerate about the physical system (Ref. 1). Thus, the existence of optimal control conveys a very important physical significance. In this report, some general existence conditions for optimal control are discussed (see Refs. 2 through 7). If the solution to the optimal-control problem exists, then it will satisfy certain necessary conditions. Loosely speaking, solutions satisfying these boundary conditions are called extremal.

However, there is no guarantee that any extremal is an optimal solution. For the general nonlinear system, there is no such guarantee nor sufficient condition of optimality. In recent years, the sufficiency conditions have virtually been ignored. However, some sufficient conditions of optimality with ordinary differential equation constraints have been studied by Lee (Ref. 5), Mangasarian (Ref. 8), Neustdat (Ref. 9), Lee and Markus (Ref. 10), and others (Refs. 11 through 15). That is, some conditions are presented that extract the optimal-control solution from the set of extremal controls.

In this report, we shall first discuss these sufficient conditions which guarantee an optimal solution, and also some features of multiple-cost functions (Ref. 16) as well. Next, we shall discuss the existence of optimal control. Examples are provided to illustrate the important features of each part (see Refs. 17 through 20).

II. SUFFICIENCY CONDITIONS

The system to be studied here, unless otherwise specified, is represented by Eq. (1) and the cost functional to be minimized by Eq. (2):

$$\dot{x} = A(t)x + B(t)u \tag{1}$$

where $A(t)$ and $B(t)$ are continuous $n \times n$ and $n \times m$ matrices, respectively, for each t in a given finite interval $[t_0,T]$ and $x(t_0) = x_0$. x is the system state, an n-vector, and u is the control, an m-vector, and the cost is

$$C(u) = g(x(T)) + C_0(u) \tag{2}$$

where

$$C_0(u) = \int_{t_0}^{T} [f^0(t,x) + h^0(t,u)]dt$$

and $g(x)$, $f^0(t,x)$, and $h^0(t,u)$ are all real continuous functions in all their arguments. Also, assume that $f^0(t,x)$,$h^0(t,u)$ are non-negative and convex functions for each t.

Denote $U(t_0,T)$ as the set of all measurable functions $u(t)$ on $[t_0,T]$ such that $u(t) \in \Omega$ for all t, Ω is a set which is in R^m. We write $u \in \Omega$, whenever u is a member of $U(t_0,T)$. $u \in \Omega$ is called admissible control. Also, let $K(T,x_0)$ denote the attainable set of the system given by Eq. (1). That is, given the initial state, $K(T,x_0)$ is the set of all points in R^n to which x_0 can be steered to via an admissible control.

Let

$$x^0(t) = \int_{t_0}^{t} [f^0(t,x) + h^0(t,u)]dt \tag{3}$$

and $\hat{K}(T,x_0)$ be the attainable set corresponding to the $(n + 1)$ dimensional state equation given by

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x + B(t)u, & x(t_0) &= x_0 \\ \dot{x}^0(t) &= f^0(t,x) + h^0(t,u), & x^0(t_0) &= 0 \end{aligned}\right\} \tag{4}$$

Let us denote the element of system (4) by $\hat{x}$, where $\hat{x}(t) = (x^0(t),x(t))$. Thus, $\hat{x}(0) = (0,x_0)$. In what follows, we assume the system $\hat{K}(T,x_0)$ has nonempty interior in R^{n+1}. The following definitions are also needed.

Definition 1. An admissible control $\bar{u}(t)$ on $[t_0,T]$, which steers $(0,x_0)$ to a boundary point of $\hat{K}(T,x_0)$, is called an extremal control, and the corresponding response $\bar{x}(t)$ is called an extremal response.

Definition 2. A control $v \in \Omega$ is called optimal if $C(v) \le C(u)$ for all $v \in \Omega$. For Lemma 1 and Theorem 1, which follow, assume Ω is such that $u \in \Omega$, whenever $|u(t)|^p \le ah^0(t,u(t))$, with $p > 1$ and $a > 0$.

Lemma 1

$K(T,x_0)$ corresponding to the system given by Eq. (4) is closed and convex. If in addition $\Omega \subset R^m$ is compact, then $\hat{K}(T,x_0)$ is compact and convex (Ref. 1).

Remark 1. It is a standard result to show that an optimal-control $u \in \Omega$ corresponding to the cost $C_0(u)$ exists. If $g(x)$ is convex in R^n, then an optimal control corresponding to $C(u)$ also exists (see Refs. 1, 5, and 7).

Lemma 2

A necessary and sufficient condition for the control $u^*(t)$ with the response $x^*(t)$ and the cost $C_0(u)$ to be extremal is that there exist a vector $\hat{\eta}(t) = (\eta_0, \eta(t))$ satisfying:

$$\dot{\eta}(t) = -\eta_0 \frac{\partial f^0}{\partial x}(t, x^*) - \eta(t)A(t)$$

$$\dot{\eta}_0 = 0, \ \eta_0 < 0$$

and

$$\eta_0 h^0(t, u^*) + \eta(t)B(t)u^* = \max_{u \in \Omega} \left\{ \eta_0 h^0(t, u) + \eta(t)B(t)u \right\}$$

almost everywhere (a.e.).

Remark 2. It can be shown that an optimal-control $u^*(t) \epsilon U(t, T_0)$ with response $x^*(t)$ is extremal; however, $u^*(t)$ may not be unique. If $h^0(t,u)$ is strictly convex in u for each t, then $u^*(t)$ will be unique. In fact, in such a case, any two extremal controls steering $(0, x_0)$ to the same boundary point of $K(T, x_0)$ must be equal almost everywhere. The uniqueness of an optimal control requires very stringent conditions. So it is natural to introduce another cost function $C_1(u)$ and then choose a control which minimizes $C_1(u)$ among the ones that minimize $C_0(u)$.

Theorem 1

Consider the system given by Eq. (1) with the corresponding cost

$$C(u) = g(x(T)) + \int_{t_0}^{T} [f^0(t,x) + h^0(t,u)]dt$$

Assume that $g(x) \in C^1$ and is convex in R^n.

(i) There exists a nontrivial solution $x^*(t)$, $\eta^*(t)$ of the system

$$\dot{x}^* = A(t)x^* + B(t)u^*$$

$$\dot{\eta}^* = -A + \frac{\partial f^0}{\partial x}(t,x^*)$$

such that $x^*(t_0) = x_0$, $\eta(T) = -\text{grad } g(x^*(T))$, and $u^*(t)$ is defined by

$$-h^0(t,u^*) + \eta B(t)u^* = \max_{u \in \Omega} [-h^0(t,u) + \eta B(t)u]$$

and is an optimal control with the corresponding optimal response $x^*(t)$.

(ii) If $h^0(t,u)$ is strictly convex for each t, then $u^*(t)$ is a unique optimal control.

A geometrical proof, outlined below in three steps, is presented.

(1) Consider the family of hypersurfaces $S_c : x^0 + g(x) = c$ in R^{n+1}. In Refs. 1 and 7, it is shown that there exists a unique hypersurface S_m among this family such that S_m is tangent to $K(T,x_0)$, and m is the optimal cost (see Fig. 1).

(2) Let $s \in \hat{K}(T,x_0) \cap S_m$ and π_s be the tangent hyperplane passing through s. Let $\hat{\eta}^*(T) = (-1, \eta^*(T))$ be normal to π_s at s. Also, let $u^*(t)$ be the extremal control steering $(0,x_0)$ to $s = \hat{x}^*(t) = (x^{*0}, x^*(t))$, and let $(-1, \eta^*(t))$ be such that $\eta^*(t)$ is the solution of

$$\dot{\eta}(t) = \frac{\partial f^0}{\partial x}(t,x^*) - \eta(t)A(t)$$

$$\eta^*(T) = -\text{grad } g(x^*(T))$$

By Lemma 2, $u^*(t)$ satisfies the maximal principle with the adjoint response $\eta^*(t)$. Thus, $u^*(t)$ is optimal since m is the optimal cost.

(3) If $h^0(t,u)$ is as in (ii), then $s \in \hat{K}(T,x_0) \cap S_m$ is the only point, and π_s is unique. Hence, from Remark 1, $u^*(t)$ must be unique.

<u>Example 1</u>. Consider the system given by Eq. (1) with the cost

$$C(u) = x'(T)Gx(T) + x^0(T)$$

such that G is a constant symmetric positive semidefinite matrix,

$$\dot{x}^0 = x'(s)W(s)x(s) + u'(s)P(s)u(s)$$

where W(s) and P(s) are both symmetric and continuous matrices for $s \in [t_0,T]$, and the prime of a matrix denotes its transpose. Let us further assume that W(s) is positive semidefinite and P(s) positive definite for all s. Let $u \in \Omega$ whenever

$$\int_{t_0}^{T} u'(s)P(s)u(s)ds < \infty$$

Since this problem satisfies the conditions of Theorem 1, and the conditions that $h^0(t,u) = u'(t)P(t)u(t)$ is strictly convex and $g(x) = x'Gx$ is convex and of class C^1 are satisfied, then from Remarks 1 and 2 there exists a unique optimal control $u^*(t) = P^{-1}(t)B'(t)\eta'(t)$ with a unique response $x^*(t)$ determined by

$$\dot{x}(t) = A(t)x(t) + B(t)P^{-1}(t)B'(t)\eta'(t), \; x(t_0) = x_0$$

$$\dot{\eta}(t) = -\eta(t)A(t) + \frac{\partial f^0}{\partial x}(t,x(t)) = -\eta(t)A(t) + 2x'(t)W(t), \; \eta(T) = -2x'(T)G$$

where $u^*(t)$ steers $\hat{x}^*(0) = (0,x_0)$ to the unique point at which the surface $S_m: x^0 + x'Gx = m$ is tangent to $\hat{K}(T,x_0)$ (see Fig. 1). In what follows, a sufficient condition for optimality is presented for a system described by a set of equations different from (1).

Consider the system given by

$$\begin{aligned} \dot{x}^0(t) &= f^0(t,x) + k^0(t,u), \; x^0(t_0) = (0,x^0), f^0, k^0 \in R^1 \\ \dot{x}(t) &= A(t)x + k(t,u), \; x(t_0) = x_0 \end{aligned} \tag{5}$$

where the coefficients f^0, $\partial f^0/\partial x$, k, k^0, and A are continuous in all arguments, and $f^0(t,x)$ is convex in x for each $t \in [t_0,T]$. The admissible controls u(t) are bounded on $[t_0,T]$, i.e., u(t) belongs to a nonempty restraint set $\Omega \subset R^m$.

Theorem 2

Part I. Consider the system given by Eq. (5) and a closed target set G in R^n with the cost $C_0(u) = x^0(T)$ and an admissible control u(t). Assume the existence of u*(t) with the response $\hat{x}*(t) = (x^{*0}(t), x^*(t))$, satisfying the maximal principle

$$-k^0(t,u^*) + \eta(t)k(t,u^*) \overset{a.e.}{=} \max_{u\in\Omega}\left\{-k^0(t,u) + \eta(t)k(t,u)\right\} \tag{6}$$

where η(t) is a nontrivial solution of

$$\eta(t) = \frac{\partial f^0}{\partial x}(t,x^*) - \eta(t)A(t)$$

and η(T) is the inward normal to the boundary ∂G of G. Then, u*(t) is optimal.

Part II. Assume

(a) $G = R^n$.

(b) The cost $C(u) = g(x(T)) + x^0(T)$, where g is differentiable and convex in R^n.

(c) The admissible control u* ∈ Ω satisfies the maximal principle by Eq. (6) with η(T) = -grad (x*(T)).

Conclusion. u*(t) is optimal.

The proof of the theorem relies heavily on two facts:

(i) $(\partial f^0/\partial x)\,(t,x)(w-x) \le f^0(t,x) = f^0(t,w)$, i.e., the convexity of $f^0(t,x)$ in x, for each fixed t.

(ii) If u* ∈ Ω such that the response $\hat{x}*(t) = (x^{*0}(t), x(t))$ initiates from $(0,x_0)$, then $\hat{\eta}(T)\hat{x}* \ge \hat{\eta}(T)\hat{w}$, where $\hat{\eta}(t)$ is as in Lemma 2, and $\hat{w}$ is in the set of attainability of the system given by Eq. (5) in R^{n+1} (see Ref. 7).

Example 2. Consider the controllable system $\dot{x}(t) = Ax(t) + Bu(t)$ in R^n with cost

$$C_0(u) = \int_0^T u'(s)P(s)u(s)ds$$

the target $G = \{x : \gamma(x) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1 \le 0\}$, $x(0) = x_0$, and P, Ω are as in Example 1.

From Remark 1, assuming that the set of attainability intersects the target set, then there exists an optimal-control $u^*(t)$ which is unique, since $h^0(t,u) = u'(t)P(t)u(t)$ is strictly convex in u for each fixed $t \in [t_0,T]$, and by the maximal principle $u^*(t) = P^{-1}(t)B'(t)\eta^*(t)'$, where $\eta^*(t)$ is the unique solution of

$$\dot{\eta}(t) = -\eta(t)A$$

with

$$\eta(T) = -\text{grad } \gamma(x^*(T)) = -2(x_1^*(T), \ldots, x_n^*(T))$$

and $x^*(t)$ is the solution of

$$\dot{x}(t) = Ax + BP^{-1}(t)B'\eta^*(t)$$

Note: $\partial G = \{x : \gamma(x) = 0\}$ and $\gamma \neq 0$ on ∂G. Hence $\eta(T) = -\text{grad } \gamma(x^*(T))$ is the inward normal to ∂G at $x^*(T)$.

III. MULTIPLE-COST FUNCTIONALS

Define the following system:

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x + B(t)u, & x(t_0) &= x_0 \\ \dot{x}^0(t) &= f^0(t,x) + h^0(t,u), & x^0(0) &= 0,\ x^0(t) \in R^1 \\ \dot{x}^1(t) &= f^1(t,x) + h^1(t,u), & x^1(0) &= 0, x^1(t) \in R^1 \end{aligned}\right\} \tag{7}$$

Assume that $f^0(t,x)$, $f^1(t,x) \in C^1$, $h^1(t,u) \in C^0$ are convex non-negative functions for each x and u, and G is the target set.

Let the admissibility be as in Theorem 2 with a further restriction on Ω, namely, Ω is compact and convex. Let $K(T,x_0)$ denote the set of attainability of the system given by Eq. (7). Our objective is to find $u^* \in \Omega$ with response $x^*(t)$ so that

(i) $x^*(T) \in G$.

(ii) $C_0(u^*) \le C_0(u)$ for $u \in \Omega$, with the corresponding response $x(t)$ in G at $t = T$.

(iii) $C_1(u^*) \le C_1(u)$ for all $u \in \Omega$ such that the corresponding response is in G at $t = T$ and $C_0(u) = C_0(u^*)$.

Definition 3. The saturation set $\hat{K}_s$, corresponding to set $\hat{K}(T,x_0)$, is the set of all points $(y^0,y) \in R^{n+1}$ such that there exists $(x^0,x) \in \hat{K}(T,x_0)$ with $x^0 \le y^0$ and $x = y$. Similarly the saturation set $\tilde{K}_s$, corresponding to set $\tilde{K}(T,x_0)$, is the set of all points $(y^0,y^1,y) \in R^{n+2}$ such that there exists a point (x^0,x^1,x) $\tilde{K}(T,x_0)$ with $x = y$, $x^0 \le y^0$, and $x^1 \le y^1$.

The following observations are made: $\tilde{K}(T,x_0)$ and $\hat{K}(T,x_0)$ are convex and compact. It is not hard to show that $\hat{K}(T,x_0)$ and $\hat{K}_s$ are the orthogonal projections of $\tilde{K}(T,x_0)$ and $\tilde{K}_s$ on (x^0,x)-space in R^{n+1}, and $K(T,x_0)$ is the orthogonal projection of $\hat{K}(T,x_0)$, $\hat{K}_s$, and $\tilde{K}(T,x_0)$ on the x-space R^n.

Theorem 3.

Assume that there exists an admissible control $u \in \Omega$ which steers the response of the system given by Eq. (1) to G at $t = T$. Then there exists a nontrivial solution $x^*(t)$, $\eta^*(t)$, $q^*(t)$ of the system of equations

$$\dot{x}(t) = A(t)x + B(t)u^*, \quad x(t_0) = x_0$$

$$\dot{q}(t) = -q(t)A(t) - \eta^0 \frac{\partial f^0}{\partial x}(t,x)$$

$$\dot{\eta}(t) = -\eta(t)A(t) - \eta^0 \frac{\partial f^0}{\partial x}(t,x) - \eta^1 \frac{\partial f^0}{\partial x}(t,x)$$

with either

(i) $x(T) \in G$, $\eta^0 < 0$, $\eta^1 < 0$, $q(T) = \eta(T) = 0$, or

(ii) $x(T) \in$ boundary of G, $\eta^0 \le 0$, $\eta^1 \le 0$, $q(T) = \eta(T)$, where $\eta(T)$ is the inward normal to G at $x(T)$, and

$$\eta^0 h^0(t,u^*) + q(t)B(t)u^* = \max_{u \in \Omega} \{\eta^0 h^0(t,u) + q(t)B(t)u\}$$

$$\eta^0 h^0(t,u^*) + \eta^1 h^1(t,u) + \eta(t)u^*B(t) = \max_{u\in\Omega} \{\eta^0 h^0(t,u) + \eta^1 h^1(t,u) + \eta(t)B(t)u\}$$

If η^0 and η^1 are non-zero, then $u^*(t)$ is an optimal control with responses $x^*(t)$, $\eta^*(t)$, and $q^*(t)$.

The proof of Theorem 3 is based on Lemmas 3 and 4.

Lemma 3

A necessary and sufficient condition for the control $u^* \in \Omega$ to steer $(0,0,x_0)$ with the corresponding response $\tilde{x}(t) = (x^0(t), x^1(t), x(t))$ to the common boundary of $\tilde{K}(T,x_0) \cap \tilde{K}_s$ at $t = T$ is that there exist a nontrivial response $\tilde{\eta}(t) = (\eta^0, \eta^1, \eta(t))$ of the equation

$$\dot{\eta}(t) = -\eta(t)A(t) - \eta^0 \frac{\partial f^0}{\partial x}(t,x) - \eta^1 \frac{\partial f^1}{\partial x}(t,x)$$

with η^0, η^1 both non-negative constants, such that

$$\eta^0 h^0(t,u^*) + \eta^1 h^1(t,u^*) + \eta(t)B(t)u^* = \max_{u\in\Omega} \{\eta^0 h^0(t,u) + \eta^1 h^1(t,u) + \eta(t)B(t)u\}$$

The proof of Lemma 3 is similar to Lemma 2 (also see Refs. 1 and 16).

Remark 3. In case $h^0(t,u)$ is strictly convex in u for each t, then Theorem 4 is reduced to a single cost function $C_0(u)$ of the system given by Eq. (1).

Lemma 4

If $K(T,x_0) \cap G$ is nonempty, then the existence of an optimal control is guaranteed, i.e., if there exists an admissible control u in Ω steering $x(t)$ to $x(T) \in G$, then there exists an optimal control. In Remark 1, the existence is always guaranteed since $K(T,x_0) \cap G$ is always nonempty.

The following example shows the applicability of Lemma 4 with a single cost function.

Example 3. Consider the linear system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

with the cost

$$C_0(u) = \int_0^t dt = t < \infty$$

Let $\Omega = \{u : |u| \leq 1\} \subset R^1$, the initial set be a fixed point

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

and $G = \{x : x_1 = 0, |x_2| \leq 1\}$, then it can be shown that $K(T, x_0) \cap G \neq \emptyset$, implying that such an initial condition in R^2 can be transferred to the set G with an optimal control.

IV. EXISTENCE OF AN OPTIMAL CONTROL

So far the existence of an optimal control for linear systems with convex integral cost has not required many stringent conditions; for nonlinear systems this is no longer true. In linear systems we have a lot of nice properties, e.g., a control function u(t) satisfies the maximal principle if and only if it is extremal. This fact is no longer true for nonlinear systems. In Ref. 1, a few examples to this effect are supplied.

In order to discuss an existence theorem for a moving target with a general cost, we shall impose the following stringent conditions:

(1) The nonlinear process $\dot{x}(t) = f(x(t), u(t), t)$, where $f \in C^1(R^{n+m+1})$.

(2) Initial and target sets $G_0(t)$ and $G_1(t)$, respectively, are compact nonempty sets and continuously varying in R^n for all $t \in (\tau_0, T)$.

(3) A control restraint Ω is a nonempty compact set continuously varying in R^m, for $(x,t) \in R^n \times (\tau_0, T)$.

(4) Inequality state constraints $h^1(x) \leq 0, \ldots, h^r(x) \leq 0$, where $h^i(x)$, $i = 1,2, \ldots, r$, is real and continuous on R^n.

(5) A nonempty family F consists of measurable functions $u(t)$ on various time intervals $(t_0,t_1) \subset (\tau_0,T)$ with response $x(t)$ steering $x(t_0) \in G(t_0)$ to $x(t_1) \in G(t_1)$, and $u(t) \in \Omega$, with condition (4) satisfied.

Theorem 4

Let $u \in F$, with the corresponding cost of the system given by

$$C(u) = g(x(t_0)) + \int_{t_0}^{t_1} f^0(x(t),u(t),t)dt + \max_{t\in(t_0,t_1)} \gamma(x(t))$$

where $f^0 \in C^1$, g and $\gamma \in C^0$ and are all real functions. Along with conditions (1)-(5) above, we impose the following additional assumptions:

(a) There exists a uniform bound $|x(t)| \leq b$ on $t \in (t_0,t_1)$ for all responses to controls $u \in F$.

(b) The velocity set $\hat{V}(x,t) = \{f^0(x(t),u(t),t), f(x(t),u(t),t) : u \in \Omega\}$ is convex in R^{n+1} for each fixed (x,t).

Then there exists an optimal control $u^*(t)$ on $[t_0^*,t_1^*] \subset [\tau_0,T]$ in F minimizing $C(u)$.

The proof relies very heavily on establishing the fact that the set of attainability corresponding to the system is compact and varies continuously in R^n, for $t \in [\tau_0,T]$. It can be shown (Ref. 10) th. F is weakly compact, implying the existence of a control $u^*(t)$ and an admissible sequence $u^k(t)$ with corresponding decreasing costs, such that

$$\lim_{n\to\infty} C(u^n) = C(u^*)$$

It is important to notice that if the initial condition t_0^* in $[\tau_0,T]$ is fixed, and F (the corresponding admissible class on various intervals $[t_0^*,T_1]$ in $[t_0^*,T]$) is nonempty, and the rest of the conditions in Theorem

4 are satisfied, then there would exist an optimal control $u^*(t)$ on $[t_0^*, t_1^*]$ in F minimizing $C(u)$ among all $u \in F$.

Remarks

(i) If $f^0 = g = 0$, then we shall have a minimax problem.

(ii) In Ref. 1 and Theorem 4, the convexity of the velocity set plays a crucial role, but in Ref. 9 several examples are given where the velocity set is not convex and the optimal control exists.

Now let us give a few examples with respect to Theorem 4.

Example 4. Consider the system:

$$\begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = -x_1 u \end{cases}$$

in R^2, $m = 1$, $x(0) = (-1,0)$ with a cost

$$C_0(u) = \int_0^{t_1} \frac{1}{1 + x_2^2}\, dt = \int_{-1}^{1} \frac{dx}{1 + y^2}$$

an admissible family F with restraint set $\Omega = \{u : |u| \le 1\}$ and $t \in [0,2]$, and $x(t_1) = (1,0)$.

For every admissible $u \in \Omega$, and $[t_0, T] = [0,2]$, there exists a response $x_1(t) = t - 1$ and $x_2(t)$, and if we let $|u(t)| = 1$, $x_2(x_1) = x_2(t(x_1))$, then we will obtain

$$\frac{-(1 - x_1^2)}{2} \le x_2(x_1) \le \frac{1 - x_1^2}{2}$$

on $x \in [-1,1]$. Since the hypotheses of Theorem 4 are satisfied, then there exists an optimal control (in fact, there are two optimal controls: $u(t) = 1$, and $u(t) = -1$).

Example 5. Let $\dot{x}_1 = \sin 2\pi u$, $\dot{x}_2 = \cos 2\pi u$, $\dot{x}_3 = -1$ in R^3 be a system with the cost

$$C_0(u) = \int_0^{t_1} (x_1^2 + x_2^2)dt$$

Ω as in Example 4, and $x(0) = (0,0,1)$, $x(t_1) = (0,0,0)$, $0 \le t \le t_1 \le 2$, be, respectively, in G_0, G_1.

Solution. The velocity set is not convex. If $t_1 = 1$, we can construct a sequence of controls $u^k(t)$ such that

$$\left.\begin{aligned} \sin 2\pi u^k(t) &= \sin 2\pi kt \\ \cos 2\pi u^k(t) &= \cos 2\pi kt \end{aligned}\right\} \quad \text{for } k = 1,2, \ldots$$

where these piecewise continuous controls are easily made (see Refs. 1 and 7). For each admissible $u^k \in \Omega$, the corresponding response is

$$x_1^k(t) = \frac{1 - \cos 2\pi kt}{2\pi k}, \quad x_2^k(t) = \frac{\sin 2\pi kt}{2\pi k}, \quad x_3^k(t) = 1 - t$$

with $x_1^k(1) = 0$, $x_2^k(1) = 0$, $x_3^k(1) = 0$,

$$C_0(u^k) = \int_0^1 \frac{1 - \cos 2\pi kt}{2\pi^2 k^2} dt = \frac{1}{2\pi^2 k^2} \text{ and } \lim_{k \to \infty} C_0(u^k) = 0$$

But there is no optimal control on $(0,1)$, since if u^* is optimal, then

$$C(u^*) = \int_0^1 (x_1^{*2} + x_2^{*2})dt = 0$$

would imply that $x^{*1}(t) = x^{*2}(t) = 0$, which implies that $\sin 2\pi u^*(t) = \cos 2\pi u^*(t) = 0$ for almost all t. Hence, an optimal control does not exist.

Example 6. We give an example where condition (2) is not satisfied. Consider $\dot{x} = u$, where u is continuous on $(0,2)$ and $x(t_0) = 0$. Define $G_1(t)$ such that

$$G_1(t) = \begin{cases} 2, & \text{if } 0 \le t \le 1 \\ 1, & \text{if } 1 \le t \le 2 \end{cases}$$

where the control restraint is the region $-1 \leq u \leq 1$, with the cost

$$C(u) = \int_{t_0}^{t_1} (x(t) - t)^2 dt$$

Let

$$u_k(t) = \frac{k}{k+1} \text{ on } \left[0, \frac{k+1}{k}\right]$$

then

$$C(u_k) = \frac{k+1}{3k^3}$$

It is obvious $G(u_k) \to 0$ when k approaches infinity. If v is the optimal control on $[t_0^*, t_1^*]$, then we must have $C(v) = 0$; therefore, $x(t) \equiv t$. We can imply that $t_0 = 0$. Hence, no optimal control exists.

Some of the conditions of Theorem 4 can be relaxed. For example, Neustadt (Ref. 9) and Warga (Ref. 19) studied the existence of optimal controls in the absence of some convexity conditions. Also Gamkrelidze (Ref. 20) showed that if the velocity set given by condition (b) of Theorem 4 was not convex, then there would exist a sequence of controls whose limit trajectory converges to a certain curve satisfying the given boundary conditions with its corresponding cost function approaching a lower bound. This limit is called the optimal sliding regime. Thus, in Theorem 4 in the absence of convexity of condition (b), if it is possible to transfer $x_0 \epsilon G_0(t)$ to $x_1 \epsilon G_1(t)$, then there would exist a minimizing sequence of controls $u_1, u_2, \ldots, u_k$, and the trajectories $x_1(t)$, $x_2(t), \ldots, x_k(t), \ldots$ such that

$$\lim_{k \to \infty} C(u_k) = M$$

where M is the infimum of the cost functions transferring x_0 to x_1. Moreover, the sequence $x_k(t)$ converges uniformly to some limit trajectory $x(t)$.

V. CONCLUSIONS

Some existence and sufficient conditions of optimality of single- and multiple-cost functions have been discussed. It has been established that the existence and sufficiency conditions cannot be taken for granted. It has also been shown that some stringent conditions on various parameters of both linear and nonlinear systems are needed in order to extract the optimal-control solutions from the set of extremals. Finally, examples conveying the effectiveness of the conditions were constructed.

REFERENCES

1. Markus, L., and Lee, E. B., "On the Existence of Optimal Controls," _J. Basic Eng._, Vol. 84, No. 13. 1962.

2. Filippov, A. F., "On Certain Questions in the Theory of Optimal," _SIAM J. Contr._, Vol. 1, No. 1, pp. 76-84, 1962.

3. Cesari, L., "An Existence Theorem in Problems of Optimal Control," _SIAM J. Contr._, Ser. A, Vol. 3, pp. 7-22, 1965.

4. Cesari, L., "Existence Theorems for Optimal Controls of the Mayer Type," _SIAM J. Contr._, Vol. 6, No. 4, pp. 517-552, 1968.

5. Lee, E. B., "A Sufficient Condition in the Theory of Optimal Control," _SIAM J. Contr._, Vol. 1, No. 3, pp. 241-243, 1963.

6. Roxin, E., "The Existence of Optimal Controls," _Michigan Math. J._, Vol. 9, pp. 109-119, 1962.

7. Lee, E. B., and Markus, L., "Optimal Control of Nonlinear Processes," _Arch. Rational Mech. Anal._, Vol. 8, pp. 36-58, 1961.

8. Mangasarian, O. L., "Sufficient Conditions for Optimal Control of Nonlinear Systems," _SIAM J. Contr._, Vol. 4, pp. 139-152, 1966.

9. Neustdat, L. W., "The Existence of Optimal Control in the Absence of Convexity Conditions," _J. Math Anal. Appl._, Vol. 7, p. 110, 1963.

10. Lee, E. B., and Markus, L., _Foundation of Optimal Control Theory_, John Wiley, New York, 1967.

11. Neustadt, L. W., "Sufficiency Conditions and a Duality Theory for Mathematical Programming Problems in Arbitrary Spaces," in _Nonlinear Programming_, edited by J. B. Rosen, O. L. Mangasarian, and K. Ritter, Academic Press, New York, 1970.

12. Rosen, J. B., _Sufficient Conditions for Optimal Control of Convex Processes_, Technical Report CS7, Stanford University, 1964.

13. Lee, E. B., "Linear Control Problems with Isoperimetric Constraints," IEEE Trans. Automat. Contr., Vol. AC-12, pp. 87-90, 1967.

14. Russell, D., "Kuhn-Tucker Conditions in Banach Space with an Application to Control Theory," J. Math. Anal. Appl., pp. 200-212, 1966.

15. Gamkrelidze, R. V., and Kharatishivili, G. L., "Extremal Problems in Linear Topological Spaces I," J. Math. Syst. Theory, Vol. 1, No. 3, pp. 229-256, 1967.

16. Dong Hak Chyung, "Optimal System with Multiple Cost Functionals," SIAM J. Contr., Vol. 5, No. 3, pp. 345-351, 1967.

17. Markus, L., "Controllability of Nonlinear Processes," SIAM J. Math., Ser. A, Vol. 3, pp. 78-90, 1965.

18. Gamkrelidze, R. V., "Some Extremal Problems," SIAM J. Control, Vol. 3, No. 1, pp. 106-127, 1965.

19. Warga, J., "Relaxed Variation Problems," J. Math Anal. Appl., Vol. 4, pp. 111-128, 1962.

20. Gamkrelidze, R. V., "Optimal Sliding States," Sov. Math. Dokl., Vol. 3, p. 559, 1962.

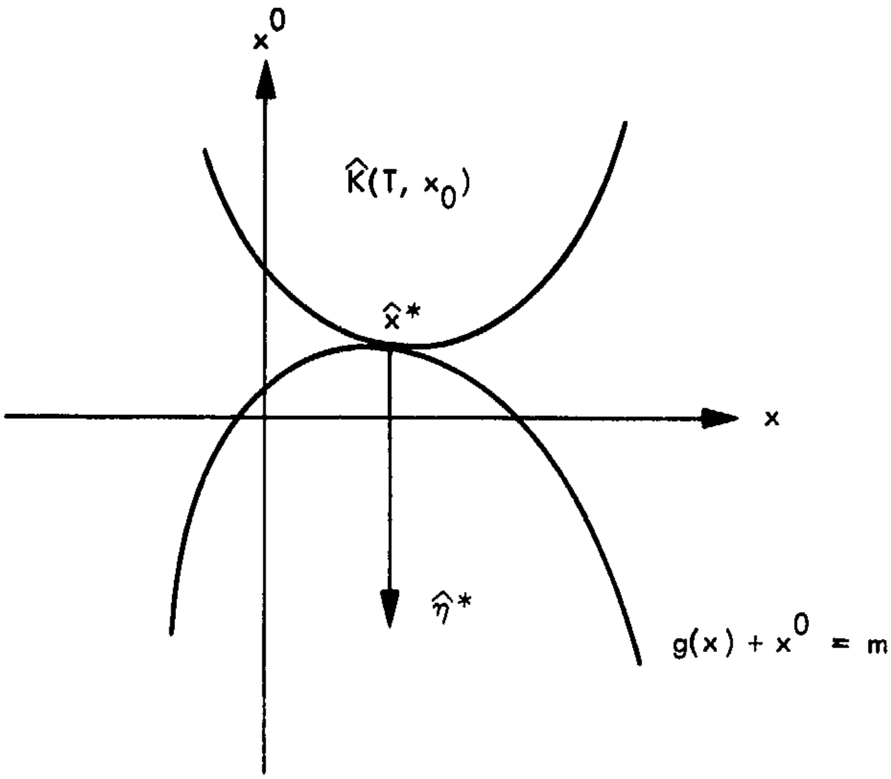

Fig. 1. Optimal hypersurface